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LETTER TO THE EDITOR

One-dimensional broken translation symmetry and pseudo-Goldstone excitation

Andrzej Radosz and Paweł Machnikowski

Institute of Physics, Wrocław University of Technology, Wybrzeże Wyspiańskiego 27, PL-50-370 Wrocław, Poland

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Abstract
In a class of potentials $U(x) = A(A\cosh^2 x - \cosh x - 2A)(1 + A\cosh x)^{-2}$ the *ground level* is $E = 0$, and the corresponding wavefunction is known exactly. This excitation is a manifestation of broken translation symmetry in one-dimensional, nonlinear models with a symmetric double-Morse potential. Broken translation symmetry in a one-dimensional model with an *asymmetric double-Morse* potential results in an exotic class of double-well potentials with exactly determined *first excited level*. Such a property, exact determination of ground or first excited level in some classes of potentials, is common for one-dimensional models with, respectively, corresponding symmetric or asymmetric double-well potentials.

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The inherent feature of one-dimensional nonlinear models with symmetric double-well potentials is the presence of kinks (localized energy excitations). Kinks, the topological excitations, are stable solutions of the equation of motion and their presence breaks continuous translation symmetry. Broken continuous internal symmetry is accompanied by the gapless Goldstone boson. That is not the case here, where the broken symmetry is continuous but is not internal. In consequence, instead of a gapless mode, there appears an isolated zero-energy pseudo-Goldstone excitation restoring the broken symmetry [1]. This pseudo-Goldstone excitation is the ground state of the corresponding 'secondary potential'. Let us illustrate this situation in the case of a 1D model defined by the Lagrangian

$$L = \int_{-\infty}^{\infty} \left[\frac{\dot{u}^2}{2} - \frac{u'^2}{2} - V(u) \right] \mathrm{d}x \tag{1}$$

with the double-Morse potential (see e.g. [2])

$$V(u) \equiv V_{\mathrm{DM}}(u) = \frac{1}{2(1 - A^2)} (A\cosh u - 1)^2 . \tag{2}$$

The kink solution of the equation of motion,

$$u_{tt} - u_{xx} + V'_{\mathrm{DM}}(u) = 0 \tag{3}$$

where prime denotes differentiation with respect to the argument of V, in this case takes the form

$$u_{\mathrm{K}}(x,t) = 2\mathrm{arctanh}\left[\sqrt{\frac{1-A}{1+A}}\tanh\left(\frac{x-vt}{2\sqrt{1-v^2}}\right)\right].$$

The analysis of the kink's stability in its rest frame

$$u(x,t) = u_{\mathrm{K}}(x) + \delta u(x)\mathrm{e}^{\mathrm{i}\omega t}$$

leads to the Schrödinger equation,

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\delta u(x) + U(x)\delta u(x) = \omega^2\delta u(x) \tag{4}$$

with the 'secondary potential'

$$U(x) \equiv V''_{\mathrm{DM}}(u_{\mathrm{K}}(x)) = A\frac{A\cosh^2 x - \cosh x - 2A}{(1+A\cosh x)^2}. \tag{5}$$

The energy of the kink solution is translation-invariant; i.e. a small shift of the kink's position

$$u_{\mathrm{K}}(x+\varepsilon) = u_{\mathrm{K}}(x) + \varepsilon u'_{\mathrm{K}}(x)$$

leaves the energy unchanged. But the kink's placement breaks this translation symmetry resulting in a pseudo-Goldstone excitation—the zero-energy eigenvalue of the secondary potential (5)

$$\omega_0^2 = 0$$

corresponding to the eigenfunction

$$\delta u_0(x) = u'_{\mathrm{K}}(x).$$

It is a common feature of all the generic symmetric double-well potentials, $V(u)$, that the secondary potential corresponding to the kink's solution of the 1D model (1)

$$\int \frac{\mathrm{d}u}{\sqrt{2V(u)}} = x - x_0$$

has the ground state

$$E_0 = 0$$

with corresponding eigenfunction

$$\eta_0(x) = u'_K(x)$$

(the symmetric double-well potential is chosen in such a way that its minima are equal to zero). The situation changes when the local symmetric double-well potential is replaced by a local asymmetric double-well potential [3] (see figure 1). There exists a bell-shaped, localized energy solution of the equation of motion for the 1D model. Such a solution is not the topological one and obviously is not a stable one, though it may be long-lived in systems where asymmetry is, in a sense, small [4, 5]. The translation symmetry is preserved in such a system but the presence of the bell shape itself breaks this symmetry resulting in a pseudo-Goldstone excitation restoring the lost symmetry. The corresponding secondary potential in the simplest case takes the shape of a double well, and at least one level of that potential, the zero-energy level, and the related eigenfunction are known exactly. Because this eigenfunction has one node it must correspond to the first excited level. In consequence, there must exist

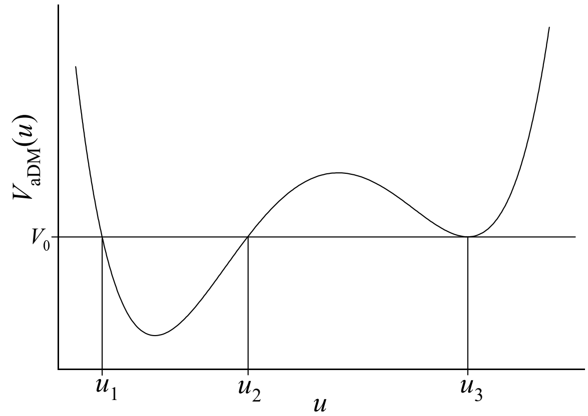

Figure 1. The asymmetric potential for $A = 0.7$, $D = 0.02$.

a negative eigenvalue of the secondary potential, responsible for the fact that the bell-shaped excitation is not a stable one. Let us illustrate the above description by using the 1D model (1) with the asymmetric double-Morse (aDM) potential

$$V_{\rm aDM} = \frac{1}{1-A^2}\left[\frac{1}{2}(A\cosh u - 1)^2 + D\sinh u\right]. \tag{6}$$

The bell-shaped solution of the equation of motion (3) with the symmetric potential (2) replaced by the asymmetric one (6) has the form

$$u_{\rm BS}(x,t) = 2\,\mathrm{arctanh}\left[z_3 - \frac{2\eta}{\beta + (z_2 - z_1)\cosh\frac{\sqrt{\eta F_0}(x-vt)}{\sqrt{1-v^2}}}\right]$$

where $z_1 = \tanh u_i/2$, with u_i defined in figure 1, and

$$F_0 = 1 + A^2 - \varepsilon^2$$
$$\beta = 2z_3 - z_2 - z_1$$
$$\eta = (z_3 - z_1)(z_3 - z_2).$$

The analysis of the bell shape's stability in its rest frame,

$$u(x,t) = u_{\rm BS}(x) + \delta u(x)\mathrm{e}^{\mathrm{i}\omega t}$$

leads to the Schrödinger equation (4), with the 'secondary potential'

$$U(x) \equiv V''(u_{\rm BS}(x)) = \frac{1}{1-A^2}\frac{A - A^2 - 2Dw - 6A^2w^2 + 2Dw^3 - (A+A^2)w^4}{(1-w^2)^2}$$

where

$$w = z_3 - \frac{2\eta}{\beta + (z_2 - z_1)\cosh\frac{\sqrt{\eta F_0}(x-vt)}{\sqrt{1-v^2}}} < 1.$$

In spite of a complicated formula, this potential has a simple double-well form as shown in figure 2.

As the bell-shaped solitary solution breaks the translation symmetry, there exists a zero-energy eigenvalue of equation (4)

$$\omega_{\rm G}(x) = 0$$

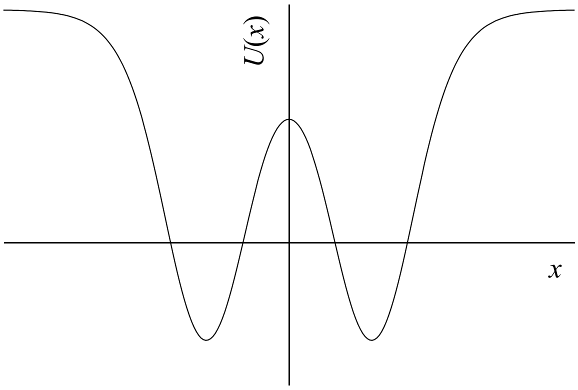

Figure 2. The 'secondary potential' related to the asymmetric double-Morse potential for $A = 0.7$, $D = 0.002$.

and the eigenfunction

$$\delta u_{\mathrm{G}}(x) = u'_{\mathrm{BS}}(x)$$

related to the pseudo-Goldstone boson. In this case the eigenfunction has one node; it therefore corresponds to the first excited level. A negative eigenvalue of the Schrödinger equation (4) must exist here, responsible for the bell-shape instability.

It is a common feature of all 1D models (1) with the generic asymmetric double-well potentials, $V(u)$, that for the secondary potential

$$U(x) \equiv V''(u_{\mathrm{BS}}(x))$$

built on the bell-shaped solution of the equation of motion (3)

$$\int_{u_0}^{u_1} \frac{\mathrm{d}u}{\sqrt{2[V(u) - V_0]}} = x - x_0$$

(V_0 being defined in figure 1), then the first excited level and the corresponding wavefunction are known exactly, namely

$$E_1 = 0$$
$$\delta u_1(x) = u'_{\mathrm{BS}}(x).$$

The ground level of the secondary potential is negative, being responsible for the instability of the bell-shaped solitary wave. The interesting feature of this asymmetric generic potential is that, introducing small asymmetry, one arrives at the long-lived bell shape and secondary potential with two very close levels ('tunnelling split'). Increasing asymmetry results in an increasing distance between the two lowest levels, where the higher of them, the pseudo-Goldstone excitation, becomes fixed, $E = 0$, and the lower one, responsible for the bell-shape lifetime, becomes shifted towards greater negative values.

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